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Deformations of topological open strings

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Abstract: Deformations of topological open string theories are described, with an emphasis on their algebraic structure. They are encoded in the mixed bulk-boundary correlators. They constitute the Hochschild complex of the open string algebra — the complex of multilinear maps on the boundary Hilbert space. This complex is known to have the structure of a Gerstenhaber algebra (Deligne theorem), which is also found in closed string theory. Generalising the case of function algebras with a B-field, we identify the algebraic operations of the bulk sector, in terms of the mixed correlators. This gives a physical realisation of the Deligne theorem. We translate to the language of certain operads (spaces of d-discs with gluing) and d-algebras, and comment on generalisations, notably to the AdS/CFT correspondence. The formalism is applied to the topological A- and B-models on the disc.

Keywords: Topological Field Theories, Chern-Simons Theories, Conformal Field Models in String Theory.

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1. Introduction

Open string theory in the presence of a constant background B-field has received much attention recently (see e.g. [1]–[4]). The effect of the 2-form B-field is to deform the algebra of functions to an algebra with a noncommutative associative star product. Kontsevich [5] has shown that associative star products are — up to a suitable gauge equivalence — in one-to-one correspondence with bivector fields, hence

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with constant 2-form background fields (the problem of deformation quantisation). Cattaneo and Felder [2] have demonstrated that the mapping effecting this correspondence has an interpretation as the perturbative expansion of the path integral of a two-dimensional topological σ-model on a disc. Furthermore, Kontsevich [6] has reframed and generalised deformation quantisation in the language of operads. These are general moulds for algebraic structures.

The primary purpose of this paper is to understand these algebraic structures in the general context of topological open string theory.

In section 2 a short review of topological closed strings is given, followed by a discussion of open string and mixed correlators, the latter intertwining between the boundary and the bulk. Using Ward identities we identify the mixed correlators with open string deformations, and using factorisation we formulate algebraic relations on them.

In section 3 we identify the algebraic (Hochschild) structure of the deformations as that of a Gerstenhaber [7]–[12].

The relations of section 2 show compatibility of these structures between the deformations of the open string theory and the bulk theory. This gives a physical realisation of the Deligne theorem, which states that the deformation theory of an associative algebra has the structure of a 'two-dimensional field theory'.

The Deligne theorem is discussed further in section 4. The structure of the deformation theory is linked to the operad of little discs, which naturally describes open and closed string diagrams.

In section 5 the formalism is applied to the topological A- and B-models. This is rather interesting and it offers plenty of scope for further research.

In section 6 we mention some directions for further work. Generalisations to higher dimensions can be made, being possibly relevant to deformations of M-theory due to a background C-field. Applications to open/closed string duality and AdS/CFT correspondence also seem to have prospects.

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2. Algebraic structure of topological open strings

In this section we describe topological open and closed string theories, emphasising the mixed correlation functions.

2.1 Review of closed strings

Topological string theory is characterised by the existence of a BRST operator Q, which is nilpotent, $Q^2 = 0$. The stress-energy tensor is BRST-exact, $T_{\mu\nu} = \{Q, G_{\mu\nu}\}$, which implies that the correlation functions of Q-closed operators are independent of the metric on the worldsheet. There is a conserved quantum number called ghost number, such that the BRST operator has ghost number 1. The Hilbert space of

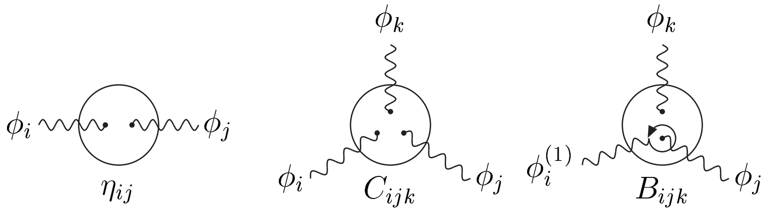

Figure 1: Closed string interactions.

topological closed string theory is spanned by operators ϕ_i. We discuss the correlation functions of these operators on the sphere. The group of global conformal transformations on the sphere is $\mathrm{SL}(2,\mathbb{C})$. Therefore, the ghost number anomaly is 6. This implies that the correlators on the sphere are nonvanishing only if the total ghost number of the operators is 6.

The two-point functions define a metric on the space of operators,

$$\langle \phi_i \phi_j \rangle = \eta_{ij} \,, \tag{2.1}$$

the three-point functions define 'structure constants',

$$\langle \phi_i \phi_j \phi_k \rangle = C_{ijk} \,. \tag{2.2}$$

The operators form an algebra, using the commutative OPE product

$$\phi_i \cdot \phi_j \sim C_{ij}{}^k \phi_k \,, \tag{2.3}$$

where $C_{ij}{}^k = C_{ijl}\eta^{lk}$. Furthermore there are three-point functions

$$B_{ijk} = \left\langle \oint \phi_i^{(1)} \phi_j \phi_k \right\rangle . \tag{2.4}$$

where the contour integral goes around the insertion point of ϕ_j.[1] They define a bracket on the space of closed string operators

$$[\phi_i, \phi_j] = B_{ij}{}^k \phi_k \,, \tag{2.5}$$

whose antisymmetry follows from the Ward identity.

One can construct a scalar 'superfield', whose highest component is denoted ϕ (a worldsheet scalar); the other components are the descendants, denoted $\phi^{(1)}$ (a 1-form) and $\phi^{(2)}$ (a 2-form). The components of the superfield satisfy the descent equations,

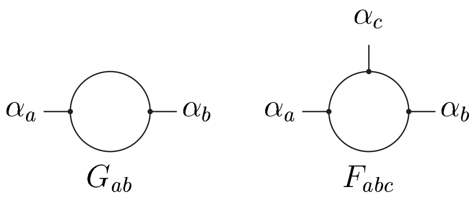

Figure 2: Open string interactions.

$$\begin{aligned} \{Q, \phi\} &= 0 \,, \\ \{Q, \phi^{(1)}\} &= d\phi \,, \\ \{Q, \phi^{(2)}\} &= d\phi^{(1)} \,. \end{aligned} \tag{2.6}$$

[1]In this paper we use the notation that a contour integral runs around the insertion point of the operator right behind it. Note that for physical operators, having ghost number 2, these correlators vanish.

There are three types of observables, given by $\phi(P)$, $\oint_C \phi^{(1)}$, and $\int_\Sigma \phi^{(2)}$, where P is a point, C is a contour and Σ is the worldsheet surface. The first operators, inserted at a point, are local operators, while the last type, as they are integrated over the worldsheet, can be used to deform the action. The second type of operators are special for two-dimensional local field theories. They can act on local operators, by choosing the contour C to enclose the insertion point P of the other operator.

2.2 Correlators on the disc

On a worldsheet with boundary (disc) we have to consider the simultaneous presence of closed string operators ϕ in the bulk and open string operators α on the boundary. For the disc, the global conformal group is $\mathrm{SL}(2,\mathbb{R})$, hence the ghost anomaly is 3. The correlators for the closed string operators are similar to the ones on the sphere discussed above.

Then we consider correlators of boundary operators. The two-point functions,

$$g_{ab} = \langle \alpha_a \alpha_b \rangle \,, \tag{2.7}$$

define a metric on the open string operators. The three-point functions,

$$F_{abc} = \langle \alpha_a \alpha_b \alpha_c \rangle \,, \tag{2.8}$$

define the structure constants of an algebra, the product of which we denote by $*$,

$$\alpha_a * \alpha_b = F_{ab}{}^c \alpha_c \,, \tag{2.9}$$

which is associative only on cohomology. Unlike in the closed string algebra, the structure constants are not required to be symmetric, but only cyclicly symmetric.

Finally, one has mixed correlators. First, there is a two-point function

$$\Phi_{ia} = \langle \phi_i \alpha_a \rangle \,, \tag{2.10}$$

defining a map from the open to the closed string algebra, if we use the closed string metric η_{ij}. Ghost counting guarantees that the correlators discussed so far are all the correlators involving only scalars.

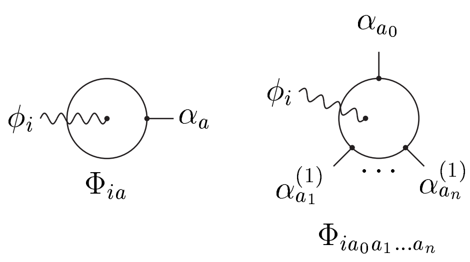

Figure 3: Mixed interactions.

Next we insert integrated second descendants of bulk operators. Let $C_{ijk,l\dots}$ denote the closed string correlator made from the three-point function by inserting one additional second descendant $\int\phi_l^{(2)}$. The Ward identity for the operator G shows that these correlators are symmetric in all indices (WDVV equations) [13, 14]. This implies that they are derivatives, e.g. $C_{ijk,l} = \partial_l C_{ijk}$, with respect to certain formal parameters. Moreover, even the three-point functions are third derivatives of some function. Introducing coupling constants t^i for each of the operators $\int\phi_i^{(2)}$, the symmetry allows us to define deformed closed string three-point functions by

$$C_{ijk}(t) = \langle \phi_i\phi_j\phi_k \mathrm{e}^{t^l\int\phi_l^{(2)}}\rangle\,. \tag{2.11}$$

In terms of these parameters, we can identify $\partial_l = \partial/\partial t^l$. In the deformed correlators, the extra insertions $t^l\int\phi_l$ can be viewed as a deformation of the bulk action.

Similarly, we introduce deformed open string three-point functions by

$$F_{abc}(t) = \langle \alpha_a\alpha_b\alpha_c \mathrm{e}^{t^l\int\phi_l^{(2)}}\rangle\,. \tag{2.12}$$

Derivatives with respect to t again introduce extra bulk operators in the correlators. The parameters t in this way deform the algebraic structure of the open string theory.

Open string three-point functions deformed by open string insertions should naively be defined by

$$\tilde{F}_{abc}(s) = \langle \alpha_a\alpha_b\alpha_c \mathrm{e}^{s^l\int\alpha_l^{(1)}}\rangle\,. \tag{2.13}$$

However, because of the ordering of operators on the boundary, this expression is not completely well-defined. Moreover, unlike in the closed string case, it is not possible to calculate open string correlators as derivatives of open-string three-point functions. For example, one has[2]

$$\partial_d\tilde{F}_{abc}(s=0) = \left\langle \alpha_a\alpha_b\alpha_c\int_c^a \alpha_d^{(1)}\right\rangle + \left\langle \alpha_a\alpha_b\int_b^c \alpha_d^{(1)}\alpha_c\right\rangle + \left\langle \alpha_a\int_a^b \alpha_d^{(1)}\alpha_b\alpha_c\right\rangle. \tag{2.14}$$

The right hand side is well-defined, and it is the variation of (2.13), whatever the precise definition of the ordering.

The separate higher-order open string correlators, defined by

$$F_{a_0a_1\dots a_n} = \left\langle \alpha_{a_0}\alpha_{a_1}\alpha_{a_2}\int\alpha_{a_3}^{(1)}\cdots\int\alpha_{a_n}^{(1)}\right\rangle, \tag{2.15}$$

provide information which is in the A_∞-structure of the open string theory (more about this below), and not in the associative structure. It is possible to choose an open string basis for which only the (undeformed) three-point functions are nonzero;

[2]Here and in the following integration of boundary operators will run between the insertion points of the neighbouring boundary operators. So the order of the boundary operators in the notation of the correlators always reflects the order of the operators on the boundary.

this corresponds to an associative structure. However, when the open string theory is deformed this is not necessarily true, so that their deformed versions $F_{a_0 a \dots a_n}(t)$ do not vanish in general [15]. The Ward identity for G shows that these correlators are cyclic in the open string indices, rather than being symmetric. Similarly, higher-order mixed correlators

$$\Phi_{ia_0 a_1 \dots a_n} = \left\langle \phi_i \alpha_{a_0} \int \alpha_{a_1}^{(1)} \cdots \int \alpha_{a_n}^{(1)} \right\rangle, \tag{2.16}$$

are cyclic in the open string indices.

When we introduce extra closed string operators in the Φ's,

$$\Phi_{ija_0 a_1 \dots a_n} = \left\langle \phi_i \int \phi_j^{(2)} \alpha_{a_0} \int \alpha_{a_1}^{(1)} \cdots \int \alpha_{a_n}^{(1)} \right\rangle, \tag{2.17}$$

we again find symmetry in i and j (see appendix A). As this is also true for the fully deformed correlators, satisfying $\Phi_{ija_0 \dots a_n} = \partial_j \Phi_{ia_0 \dots a_n}$, they are integrable: there are functions $\Phi_{a_0 \dots a_n}$ such that $\Phi_{ia_0 \dots a_n} = \partial_i \Phi_{a_0 \dots a_n}$.

We can however go further, and show that the mixed correlators are related to deformations of the open string algebra. The derivation, which is in appendix A, is similar to the derivation of the WDVV equations [13]. The result is

$$\left\langle \int \phi_i^{(2)} \alpha_a \alpha_b \alpha_c \right\rangle = \left\langle \phi_i \alpha_a \int \alpha_b^{(1)} \int \alpha_c^{(1)} \right\rangle, \quad \text{i.e.} \quad \partial_i F_{abc} = \Phi_{iabc}\,. \tag{2.18}$$

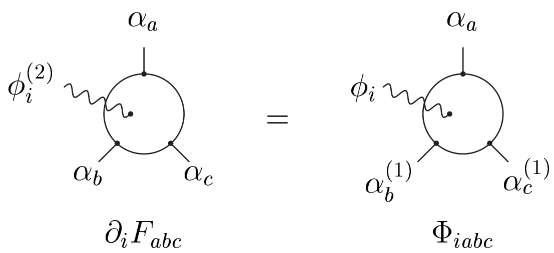

Figure 4: Deformation of the product.

This expresses that the element Φ_{iabc} is a deformation of the product by closed string operators, see figure 4.

Upon inserting extra operators $\int \alpha_b^{(1)}$, we obtain $\partial_i F_{a_0 \dots a_n} = \Phi_{ia_0 \dots a_n}$. This shows again that the correlators $\Phi_{ia_0 \dots a_n}$ are derivatives, and it provides a way to calculate the functions $\Phi_{a_0 \dots a_n}$.

We want to view the mixed correlators as intertwiners between the closed string algebra and the deformations of the A_∞ structure, given by the boundary correlators

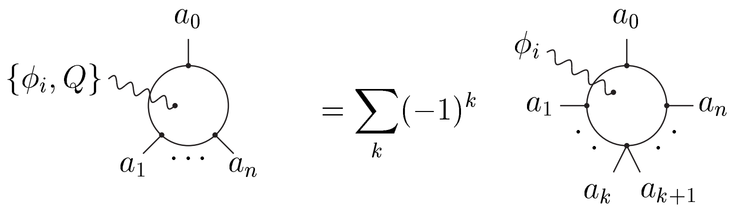

Figure 5: Factorisation of the BRST operator.

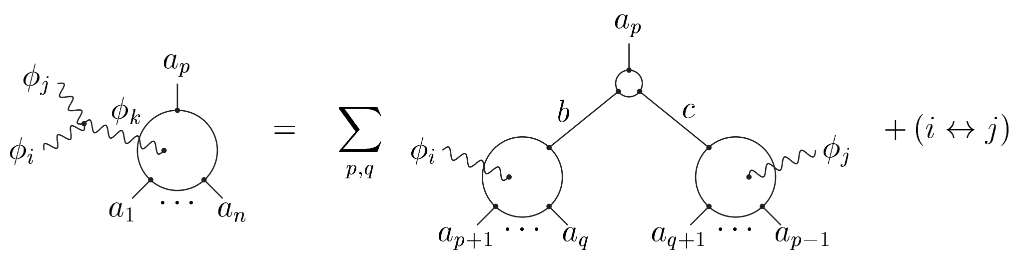

Figure 6: Factorisation of the product.

$F_{abc\ldots}$. An essential structure of the topological bulk theory is the BRST operator. A BRST operator acting on the closed string operator in the mixed correlators can be deformed to a contour around the boundary operators. Using the descent equations for the boundary operators gives the following identity, also depicted in figure 5.

$$\begin{aligned}\Big\langle \{\phi_i, Q\}\alpha_{a_0} \int \alpha^{(1)}_{a_1} \cdots \int \alpha^{(1)}_{a_n} \Big\rangle &= \Big\langle \phi_i(\alpha_{a_0} * \alpha_{a_1}) \int \alpha^{(1)}_{a_2} \cdots \int \alpha^{(1)}_{a_n} \Big\rangle + \\ &+(-1)^{n+1}\Big\langle \phi_i \int \alpha^{(1)}_{a_1} \cdots \int \alpha^{(1)}_{a_{n-1}} (\alpha_{a_n} * \alpha_{a_0}) \Big\rangle + \\ &+\sum_{k=1}^{n-1} (-1)^k \Big\langle \phi_i \alpha_{a_0} \int \alpha^{(1)}_{a_1} \cdots \int (\alpha_{a_k} * \alpha_{a_{k+1}})^{(1)} \cdots \\ &\qquad \cdots \int \alpha^{(1)}_{a_n} \Big\rangle. \end{aligned} \tag{2.19}$$

In this derivation, the boundary operators are taken on-shell (BRST-closed), while for ϕ_i we take an arbitrary local closed string operator. The boundary operators are assumed to have odd ghost degree; otherwise extra signs are introduced.

We will see that the factorisation of certain diagrams gives the algebra of mixed correlators the full structure of a closed string algebra. First, the closed string metric η_{ij} factorises in the open string channel,

$$\eta_{ij} = \Phi_{ia} g^{ab} \Phi_{jb}\,. \tag{2.20}$$

There is also a straightforward factorisation of the $(n+2)$-point functions with two closed string scalars:

$$\Big\langle \phi_i \phi_j \alpha_b \int \alpha^{(1)}_{a_1} \cdots \int \alpha^{(1)}_{a_n} \Big\rangle\,. \tag{2.21}$$

This gives the following identity:

$$C_{ij}{}^k \Phi_{ka_0\ldots a_n} = \sum_{p,q=0}^{n} (-1)^{p+q-1} F_{bca_p} \Phi_i{}^b{}_{a_{p+1}\ldots a_q} \Phi_j{}^c{}_{a_{q+1}\ldots a_{p-1}} + (i \leftrightarrow j)\,. \tag{2.22}$$

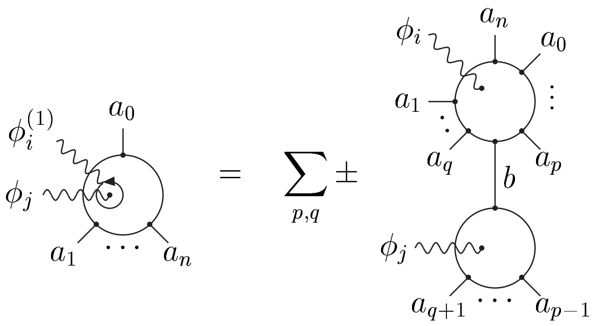

Figure 7: Factorisation of the bracket.

This is a commutative product on the algebra of physical operators, depicted in figure 6. Notice that the integrated open string operators can in principle also be inserted at the open string three-point interaction, but as remarked before, this amounts to considering the A_∞ rather than the associative structure. That, in turn, corresponds to the definition (3.6) of the product of the Gerstenhaber algebra acquiring higher-order corrections to the bilinear $*$ operation.

Finally, factorisation of the bracket,

$$\left\langle \alpha_{a_0} \int \alpha_{a_1}^{(1)} \cdots \int \alpha_{a_n}^{(1)} \oint \phi_i^{(1)} \phi_j \right\rangle , \tag{2.23}$$

gives the identity

$$B_{ij}{}^k \Phi_{k a_0 a_1 \ldots a_n} = \sum_{p,q} (-1)^{p+q-1} \Phi_{i b a_p \ldots a_q} \Phi_j{}^b{}_{a_{q+1} \ldots a_{p-1}} , \tag{2.24}$$

depicted in figure 7. We will prove this identity in the next section.

3. Deformations and Hochschild complex

The above structure of the BRST operator, product and bracket will be interpreted as a natural algebraic structure on the deformations of the open string theory.

3.1 String theories and algebras up to homotopy

It has been shown that open string theory has the structure of a homotopy associative, or A_∞-algebra [16]–[19] (see appendix B). This is a generalisation of associative algebra, where the associativity constraint is replaced by an infinite number of similar relations. We saw this reflected in the fact that due to the ordering on the boundary, higher-point functions cannot be written as derivatives, cf. eq. (2.14).

It has also been recognised that (topological) closed string theory is related to homotopy Lie algebras, and with extra structure, to (homotopy) Gerstenhaber algebras, see e.g. [7]–[12]. A Gerstenhaber algebra is an associative algebra with a symmetric product (eq. (2.3)), which in addition is provided with a graded Lie bracket (eq. (2.5)) of (ghost) degree -1, satisfying certain associativity and compatibility relations (see appendix B).

More familiar in string theory is the notion of a BV algebra, which is a Gerstenhaber algebra whose bracket can be expressed in terms of a BV operator Δ of ghost number -1, squaring to 0. It is not a derivation of the product though, as the failure of the Leibniz rule determines the bracket,[3]

$$\{\phi_i, \phi_j\} = \Delta(\phi_i \cdot \phi_j) - \Delta\phi_i \cdot \phi_j \mp \phi_i \cdot \Delta\phi_j \,. \tag{3.1}$$

A generalisation of the claims in the present paper to deformations of closed strings would require a closer look at the differences between Gerstenhaber and BV structures.

Gerstenhaber and BV algebras can be extended to homotopy Gerstenhaber and homotopy BV algebras. Physically, the higher homotopies correspond to BRST-exact terms; passing to BRST cohomology eliminates them.

3.2 Gerstenhaber structure

How an algebra A is deformed is determined by its Hochschild complex Hoch(A). Let us first focus on graded A_∞-algebras A, such as open string algebras. Besides the ordinary bilinear operation (product), these can have n-linear operations, of degree $2-n$, satisfying the master equation (see appendix B).

For any algebraic structure, the relations and symmetries can be formulated in terms of multilinear maps. For example, the structure constants $F^{a_0}{}_{a_1\ldots a_n}$ of the A_∞ algebra of the boundary operators define an n-linear map on the space of these operators. Also the deformations of the algebraic structure are multilinear maps. We found that the deformations of the structure constants are encoded in the mixed correlation functions (2.16). In fact, for a given closed string operator ϕ_i, we may identify the object in (2.16) as a map $\Phi : A^{\otimes n} \to A$, using the open string metric g_{ab}:

$$\Phi_i(\alpha_{a_1}, \ldots, \alpha_{a_n}) = \Big\langle \phi_i \alpha_b \int \alpha^{(1)}_{a_1} \cdots \int \alpha^{(1)}_{a_n} \Big\rangle g^{bc} \alpha_c = \Phi_{iba_1\ldots a_n} \alpha^b = \partial_i F_{ba_1\ldots a_n} \alpha^b, \tag{3.2}$$

with inverse relation

$$\Phi_{iba_1\ldots a_n} = \langle \alpha_b \, \Phi_i(\alpha_{a_1}, \ldots, \alpha_{a_n}) \rangle \,. \tag{3.3}$$

The last equality in (3.2) shows that we can interpret the maps Φ_i as infinitesimal deformations of the higher structure constants $F^a{}_{bc\ldots}$ of the A_∞ algebra.

The vector spaces $C^n(A, A) = \mathrm{Hom}(A^{\otimes n}, A)$, consisting of n-linear maps in A, define the degree n space of the Hochschild complex Hoch(A) of the algebra A. Using the closed string metric η_{ij} as well, one can equivalently view Φ as a map from

[3]For the bosonic string, the BV operator is given by the anti-ghost zero-mode $b_0^- = b_0 - \bar{b}_0$.

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$\mathrm{Hom}(A^{\otimes n}, A)$ to the closed string algebra of the ϕ_i. In this way, we have canonical maps between the Hochschild complex of the open string algebra and the space of closed string operators. We will usually identify the correlator with the corresponding mapping.

The Hochschild complex of a (homotopy) associative algebra is naturally endowed with three operations: a coboundary operator (of degree 1), a product (of degree 0) and a bracket (of degree -1).

- The Hochschild coboundary operator δ acts by contraction and by adding two boundary terms. For $\Phi \in C^n(A, A)$ it is defined by

$$(\delta\Phi)(\alpha_1, \ldots, \alpha_{n+1}) = \alpha_1 * \Phi(\alpha_2, \ldots, \alpha_{n+1}) + (-1)^{n+1}\Phi(\alpha_1, \ldots, \alpha_n) * \alpha_{n+1} + \\ + \sum_{k=1}^{n}(-1)^k \Phi(\alpha_1, \ldots, \alpha_k * \alpha_{k+1}, \ldots, \alpha_{n+1}) . \tag{3.4}$$

  Equation (2.19) expresses that this Hochschild coboundary operator equals the BRST operator Q of the closed string,

$$\delta\Phi(\alpha_{a_1}, \ldots, \alpha_{a_{n+1}}) = \Big\langle \{\Phi, Q\}\alpha_b \int \alpha^{(1)}_{a_1} \cdots \int \alpha^{(1)}_{a_{n+1}} \Big\rangle \alpha^b . \tag{3.5}$$

- The product is the cup product of two elements $\Phi_i \in C^{n_i}(A, A)$, defined by

$$(\Phi_1 \cup \Phi_2)(\alpha_1, \ldots, \alpha_{n_1+n_2}) = (-1)^{n_1 n_2}\Phi_1(\alpha_1, \ldots, \alpha_{n_1}) * \Phi_2(\alpha_{n_1+1}, \ldots, \alpha_{n_1+n_2}) . \tag{3.6}$$

  It is associative, but graded commutative only on cohomology. In deformed topological open string theory it is given by (2.22).

- The (twisted) Lie bracket (the Gerstenhaber bracket) for $\Phi_i \in C^{n_i}(A, A)$ is given by

$$\{\Phi_1, \Phi_2\} = \Phi_1 \circ \Phi_2 - (-1)^{(n_1-1)(n_2-1)}\Phi_2 \circ \Phi_1 , \tag{3.7}$$

  where $\circ$ is the composition of mappings on complexes,

$$(\Phi_1 \circ \Phi_2)(\alpha_1, \ldots, \alpha_{n_1+n_2-1}) = \tag{3.8}$$
$$= \sum_{k=1}^{n_1+n_2-1} (-1)^{k(n_2-1)}\Phi_1(\alpha_1, \ldots, \alpha_k, \Phi_2(\alpha_{k+1}, \ldots, \alpha_{k+n_2}), \alpha_{k+n_2+1}, \ldots, \alpha_{n_1+n_2-1}) .$$

  We mentioned, and we will prove below that in deformed topological open string theory it is given by the closed string bracket (2.24).

The Hochschild coboundary operator and the Gerstenhaber bracket give $C^*(A, A)$ the structure of a differential graded Lie algebra, with the cup product it is also a differential Gerstenhaber algebra. The master equation for the Gerstenhaber structure says that Q is a nilpotent derivation for the product and the bracket, and that the bracket satisfies the Jacobi identity up to BRST-exact terms (cf. [10]). These terms correspond to higher homotopies in the Gerstenhaber algebra.

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3.3 Maurer-Cartan equation and bracket

The Hochschild complex shifted by 1 of an associative algebra can be regarded as its deformation complex. The associativity of the product can be stated in the A_∞ language as $d_2^2 = 0$. The objects which deform A are in its Hochschild cohomology. The Φ-deformed product can be written $d_2'(\alpha_a, \alpha_b) = d_2(\alpha_a, \alpha_b) + \Phi(\alpha_a, \alpha_b)$, where $d_2(\alpha_a, \alpha_b) = \alpha_a * \alpha_b$ and

$$\Phi(\alpha_a, \alpha_b) = (F_{ab}{}^c(t) - F_{ab}{}^c(0))\alpha_c = t^i \Phi_{iab}{}^c \alpha_c + \frac{1}{2} t^i t^j \Phi_{ijab}{}^c \alpha_c + \mathcal{O}(t^3)\,. \tag{3.9}$$

Figure 4 and eq. (2.18) show the deformation caused by $\phi_i^{(2)}$ as a Hochschild cocycle.

Preserving associativity when deforming with a Hochschild cocycle Φ is equivalent to the Maurer-Cartan equation for Φ,

$$\delta\Phi + \Phi \circ \Phi = \delta\Phi + \frac{1}{2}\{\Phi, \Phi\} = 0\,, \tag{3.10}$$

because we can write (3.4) as $d_2 \circ \Phi + \Phi \circ d_2 = \delta\Phi$. For infinitesimal deformations $\Phi = t_i \Phi_i + \mathcal{O}(t^2)$, we find $\delta\Phi = 0$, therefore Φ is BRST-closed. In fact, the infinitesimal element is in the Hochschild cohomology. The Maurer-Cartan equation can be used to find a finite deformation in terms of the infinitesimal one. For example, for a single deformation parameter $t^i = t$, we may write (3.9) as $\Phi(t) = \sum_{n\geq 1} 1/n! \Phi_{(n)} t^n$, and we find recursive relations

$$\delta\Phi_{(n)} = -\frac{1}{2} \sum_{k+l=n;\ k,l\geq 1} \{\Phi_{(k)}, \Phi_{(l)}\}\,. \tag{3.11}$$

We will show that in deformed topological open string theory, the bracket $\{\cdot, \cdot\}$ is related to the closed string bracket,

$$[\phi_i, \phi_j] = \oint \phi_i^{(1)} \phi_j\,, \tag{3.12}$$

contracted with the couplings. The Maurer-Cartan equation can be derived from the factorisation of the fully deformed four-point function on ϕ-cohomology. Without integrated boundary insertions we write, keeping only the position of α_a free,

$$\begin{aligned}
\int d\langle \alpha_a \alpha_b \alpha_c \alpha_d\, \mathrm{e}^{t^k \int \phi_k^{(2)}} \rangle &= \int \langle \{Q, \alpha_a^{(1)}\} \alpha_b \alpha_c \alpha_d\, \mathrm{e}^{t^k \int \phi_k^{(2)}} \rangle \\
&= \Big\langle \int \alpha_a^{(1)} \alpha_b \alpha_c \alpha_d \Big\{ Q, t^i \int \phi_i^{(2)} \Big\}\, \mathrm{e}^{t^k \int \phi_k^{(2)}} \Big\rangle \\
&= \Big\langle \int \alpha_a^{(1)} \alpha_b \alpha_c \alpha_d\, t^i \oint \phi_i^{(1)}\, \mathrm{e}^{t^k \int \phi_k^{(2)}} \Big\rangle \\
&= t^i t^j \Big\langle \int \alpha_a^{(1)} \int \alpha_b^{(1)} \int \alpha_c^{(1)} \alpha_d [\phi_i, \phi_j]\, \mathrm{e}^{t^k \int \phi_k^{(2)}} \Big\rangle,
\end{aligned} \tag{3.13}$$

where we used the descent equations and the fact that that the BRST operator Q and the operation $\oint\phi_i$ act as derivations on the exponential.

The left-hand side can be written as a pure boundary sum,

$$\langle\alpha_a\alpha_b\alpha^e\,\mathrm{e}^{t^k\int\phi_k^{(2)}}\rangle\langle\alpha_e\alpha_c\alpha_d\,\mathrm{e}^{t^k\int\phi_k^{(2)}}\rangle-\langle\alpha_b\alpha_c\alpha^e\,\mathrm{e}^{t^k\int\phi_k^{(2)}}\rangle\langle\alpha_e\alpha_d\alpha_a\,\mathrm{e}^{t^k\int\phi_k^{(2)}}\rangle\,. \tag{3.14}$$

Associativity of the deformed product is the vanishing of this expression. Putting (3.13) to zero is exactly the Maurer-Cartan equation (3.10), if we identify the bracket on the open string deformations with the bracket in the closed string theory.

Up to first order in t, the vanishing is guaranteed by undeformed associativity and δ-closedness of ϕ. In second order in t, we find from (3.13) and (3.14):

$$\begin{aligned}2\Big\langle\int\alpha_a^{(1)}\int\alpha_b^{(1)}\int\alpha_c^{(1)}\alpha_d[\phi_i,\phi_j]\Big\rangle=&\Big\langle\alpha_a\alpha_b\alpha^e\int\phi_i^{(2)}\int\phi_j^{(2)}\Big\rangle\langle\alpha_e\alpha_c\alpha_d\rangle+\\&+\langle\alpha_a\alpha_b\alpha^e\rangle\Big\langle\alpha_e\alpha_c\alpha_d\int\phi_i^{(2)}\int\phi_j^{(2)}\Big\rangle+\\&+\Big\langle\alpha_a\alpha_b\alpha^e\int\phi_i^{(2)}\Big\rangle\Big\langle\alpha_e\alpha_c\alpha_d\int\phi_j^{(2)}\Big\rangle\mp\\&\mp\text{cyclic perms.}\end{aligned} \tag{3.15}$$

The first two terms on the right-hand side together equal half the left-hand side. The derivation is similar to (2.19), with ϕ_i replaced by $\phi_i\int\phi_j^{(2)}$.[4] The bracket is therefore given by

$$\Big\langle\alpha_d\,[\Phi_i,\Phi_j](\alpha_a,\alpha_b,\alpha_c)\Big\rangle=\Big\langle\alpha_a\alpha_b\alpha_e\int\phi_i^{(2)}\Big\rangle\Big\langle\alpha^e\alpha_c\alpha_d\int\phi_j^{(2)}\Big\rangle\mp\text{cyclic perms.} \tag{3.16}$$

With extra boundary insertions, the derivation goes along the same lines, with both sides acquiring extra contributions. This gives (2.24). Finally we remark that the factorisation can be generalised to the higher relations of the A_∞ algebra, giving generalised Maurer-Cartan equations.

3.4 Deformation complex: abstract setting

At this point it is useful to examine the more abstract relation between deformations and Hochschild complex. The deformation algebra $\mathrm{Def}^*(A)$ of any algebra is a graded Lie algebra, formed by the possible deformations of the structure. These comprise not only the actual deformations, but also symmetries and obstructions to deformations. They always contain a trivial part equal to the algebra itself, which

[4] In general, we have the operator identity $Q\Big(\phi_i\mathrm{e}^{t^k\int\phi_k^{(2)}}\Big)=t^j[\phi_j,\phi_i]\mathrm{e}^{t^k\int\phi_k^{(2)}}$. This reflects the well-known fact that the BRST operator is deformed by the bracket with $t^j\phi_j$ [20]. The extra integral should be performed on a (regularized) disc inside the contour used for the action of the BRST operator. Therefore we do not get deformed star products at the right-hand side of (2.19), but only the first two terms in (3.15).

$\mathrm{Def}^0(A)$	Symmetry	CS
A^0	$\delta\alpha = \{Q, \lambda_0\}$	$\delta A = d_A\lambda_0$
A^1	$\delta\phi = \{Q, \lambda_1\}$	$\delta B = d_A\lambda_1$
$\mathrm{End}(A)^0$	$\delta\alpha = \lambda(\alpha)$	$\delta A = \lambda(A)$
$\mathrm{Def}^1(A)$	Deformation	CS
A^1	$\delta\alpha = \alpha_1$	$\delta A = \lambda_1$
A^2	$\delta F = \varphi_2$	$\delta F = b_2$
$\mathrm{End}(A)^1$	$\delta Q = \oint \varphi_2^{(1)}$	$\delta d_A f = a_1 \wedge f$
$\mathrm{Hom}(A^{\otimes 2}, A)^0$	$\delta * = \int \varphi_2^{(2)}$	$f\ \delta * g = \theta(f, g)$
$\mathrm{Def}^2(A)$	Obstruction	CS
A^2	$\{Q, \alpha\} = 0$	E.o.m.
A^3	$\{Q, F\} = 0$	Bianchi
$\mathrm{End}(A)^2$	$Q^2 = 0$	Derivation
$\mathrm{Hom}(A^{\otimes 2}, A)^1$	$[Q, *] = 0$	Leibniz
$\mathrm{Hom}(A^{\otimes 3}, A)^0$	$*^2 = 0$	Associativity

Table 1: Components of $\mathrm{Def}^*(A)$ and their interpretations as gauge symmetries, deformations and obstructions. δ denotes a variation. The structure of the open string algebra is determined by the operations F, Q, and $*$.

represents the shift of the origin in the space of operators. The rest consists of the various deformations of multilinear maps, contained in the Hochschild complex, with the grading shifted by 1. That is,

$$\mathrm{Def}^p(A) = A^p \oplus \mathrm{Hoch}(A)^{p+1} = A^p \oplus \bigoplus_{n\geq 0} \mathrm{Hom}(A^{\otimes n}, A)^{p+1-n} . \tag{3.17}$$

The superscript is the grading. The shift in grading by one is actually a convention, as the degree in the mathematical deformation complex is always such that the actual (physical) deformations have degree one, and the gauge symmetries degree zero. The physical ghost number is more in line with the degree in the Hochschild complex. For example, the physical deformations are contained in the Hochschild complex at degree two, which is exactly the ghost number of the physical bulk operators (in the zeroth descendants form). The extra shift of n in the last identification is related to the fact that exactly that number of boundary operators in the corresponding correlators $\Phi_{ia_0\ldots a_n}$ are descendants.

The elements of degree 0 correspond to the (gauge) symmetries of the algebra. The actual physical deformations are contained in Def^1. The degree 2 part, Def^2, consists of the obstructions to deformations. They express the conditions that should be satisfied. Physically, they correspond to classical equations of motion of the background. Some important components of $\mathrm{Def}^*(A)$ are summarised in table 1. The last column shows the interpretation in the Chern-Simons gauge

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theory, which can be viewed as an open string theory. This will be discussed in some more detail in section 5. The BRST operator is the covariant derivative. In general, after a deformation, we also have to include an element F in the algebra, which can be interpreted as a degree zero component d_0 of the total differential on the A_∞ algebra (compare appendix B). On-shell this element is clearly zero. In CS, it has the interpretation of the field strength. The deformation in $A^2 \subset \mathrm{Def}^1(A)$ by the degree two element corresponds to a deformation in the action by a shift of this element F. In the gauge theory it is similar — and actually directly related in a topological limit — to the addition of the two-form field B to the total field strength. Its symmetry is contained in $A^1 \subset \mathrm{Def}^0(A)$.

In the cohomology, the deformation $\delta\alpha = \alpha_1$ is cancelled by the (gauge)symmetry $\delta\phi_2 = \{Q, \lambda_1\}$, taking $\lambda_1 = \alpha_1$. The differential of the deformation complex maps $A^1 \subset \mathrm{Def}^0(A)$ identically onto $A^1 \subset \mathrm{Def}^1(A)$, so that in cohomology they cancel each other.

3.5 A special case

As a special case, consider $A = C^\infty(M)$, the case of Kontsevich and of Seiberg and Witten (functions or polynomials). The Hochschild complex shifted by 1 is the complex of polydifferential operators on M; the star product corresponds to a bidifferential $A^{\otimes 2} \to A$.

The Hochschild cohomology consists of the polyvector fields on M, which have an independent differential Gerstenhaber structure, given by a vanishing differential, wedge multiplication of polyvector fields, and the Schouten-Nijenhuis bracket, which is the usual Lie bracket on vector fields, and is extended to polyvector fields by the Leibniz rule, $[\theta_1, \theta_2 \wedge \theta_3] = [\theta_1, \theta_2] \wedge \theta_3 + (-1)^{|\theta_1|(|\theta_2|-1)}\theta_2 \wedge [\theta_1, \theta_3]$.

A bivector field θ satisfying the Maurer-Cartan equation on cohomology, $[\theta, \theta] = 0$, corresponds to a Poisson structure $\{f, g\} = \theta^{ij}\partial_i f \partial_j g$ on M and, when invertible, to a 2-form $B = \theta^{-1}$ satisfying $dB = 0$.[5] Indeed, B is the physical B-field, a closed string state.

In this special case, there is a correspondence between deformers (in the Hochschild cohomology) and deformations (in the Hochschild complex) preserving the algebra structure, so that the brackets, and hence the Maurer-Cartan equations, are mapped onto each other. This has been shown for the differential graded Lie structure by Kontsevich [5] and for the differential graded Gerstenhaber structure by Tamarkin [21]. The correspondence, whose existence is captured by the formality theorem, can be given by the perturbative expansion of the Cattaneo-Felder model [2].

[5] In components of θ, $\theta^{[il}\partial_l\theta^{jk]} = 0$.

4. Operad formulation and deligne theorem

The algebraic structure of the Hochschild complex is encoded in the statement that it is an algebra over the 'operad of little discs' [6]. This operad is closely related to two-dimensional CFTs.

4.1 Little discs

Operads are mathematical devices to describe multilinear operations. They are certain collections $\mathrm{O}(n)$, with actions of the symmetry group S_n acting on them. They form a useful tool to study (local) quantum field theories, as there collections of operators having a particular behaviour under interchanges are associated to a point in some topological space. This has been recognised in the literature [9, 10].

The most relevant operad for us, and in general for a local perturbative quantum field theory, is the operad C_d of little d-discs. Here d is any nonnegative integer, which stands for the (spatial) dimension. It is a topological operad consisting of collections $C_d(n)$ of n d-dimensional discs (holes) D_i, $i = 1, \ldots, n$, inside the standard disc

$$D_0 := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; x_1^2 + \ldots + x_d^2 \leq 1 \right\}. \tag{4.1}$$

There is a natural action of the symmetric group S_n on $C_d(n)$, given by the permutation of the indices of the little discs D_i. The first two collections are given by $C_d(0) = \emptyset$ and $C_d(1) = \text{point} = \mathrm{id}_{C_d}$.

On these collections of discs, there are natural compositions

$$C_d(k) \times C_d(n_1) \times \cdots C_d(n_k) \to C_d(n_1 + \cdots + n_k)\,, \tag{4.2}$$

given by gluing k discs inside a disk with k holes. Being a bit more precise, let G_d be the $(d+1)$-dimensional Lie group acting on $\mathbb{R}^d$ by affine transformations $u \mapsto \lambda u + v$ where $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^d$. The composition law is obtained by applying elements from G_d associated with discs D_i, $i = 1, \ldots, k$ in the configuration in $C_d(k)$ to configurations in all $C_d(n_i)$, $i = 1, \ldots, k$, and putting the resulting configurations together.

For $d = 2$ we can naturally relate this to the composition of closed string diagrams. The little discs D_i are the holes in which the (regularised) local operators are inserted, while the standard disc D_0 is to be thought of as the outgoing state at infinity. The affine group G_2 is the subgroup of the conformal group that remains after the choice of the standard disc.

For technical reasons, one also introduces the operad $\mathrm{Chains}(C_d(n))$ of chain complexes on the topological operad C_d. Its cohomology is the homology operad $H_*(C_d)$. For the precise definition, see [6].

4.2 d-Algebras

Next, to define a local QFT on a topological space, one has to introduce an algebra of operators, which are to be inserted in the holes. The mathematical objects related to this are d-algebras, which are related to the above operad.

An *algebra over an operad O* of vector spaces (complexes) is a vector space (complex) V provided with a morphism of operads

$$\mathrm{O}(n) \to \mathrm{Hom}(V^n, V)\,, \qquad n \geq 1\,. \tag{4.3}$$

We can also write this as a sequence of maps

$$\mathrm{O}(n) \times V^n \to V\,, \qquad n \geq 1\,, \tag{4.4}$$

satisfying certain compatibility conditions [9]. One also says that *the operad O acts on V*. For the operad of little discs, (4.4) can be thought of as a collections of holes connected to operators in a linear space V, which map to an outgoing state in V (at the fixed point at infinity).

A d-algebra is an algebra over the operad $\mathrm{Chains}(C_d)$ in the category of complexes. Because C_0 is a point, a 0-algebra is a complex. One can show that any 1-algebra carries a natural structure of an A_∞-algebra. A d-algebra can thus be characterised by a vector space V together with maps in $\mathrm{Hom}(V^{\otimes n}, V)$, possessing a natural action of the groups S_n, and endowed with composition laws compatible with the structure of the operad C_d. $\mathrm{Chains}(C_d(n))$ can be shown to be quasi-isomorphic to its cohomology, $H_*(C_d)$, endowed with zero differential.[6] d-algebras are thus essentially algebras over $H_*(C_d)$, and these can be classified. Up to homotopy, an algebra over $H_*(C_d)$ is [6]

- a complex if $d = 0$,
- a differential graded associative algebra if $d = 1$,
- a differential graded "twisted" Gerstenhaber algebra with bracket of degree $1 - d$ for $d \geq 3$ even,
- a differential graded "twisted" Poisson algebra with bracket of degree $1 - d$ for $d \geq 3$ odd.

Physically, these characterisations are quite natural. For $d = 0$ we should think of instantons; the operators for an instanton form a vector space. For a point particle, or $d = 1$, we also know that the operators in the quantum mechanics form an associative algebra. The product can be thought of as applying two consecutive perturbations at different times, and measuring the outcome at infinite time (having the topology of an interval with two holes). We mentioned already that closed string theory ($d = 2$) has the structure of a Gerstenhaber algebra.

[6] A quasi-isomorphism between objects of a given structure is a morphism with respect to this structure, inducing an isomorphism of cohomologies.

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4.3 A realisation of the deligne theorem

The Deligne theorem, proved by Kontsevich and Soibelman [30], states that there exists a natural action of the operad Chains(C_2) on the Hochschild complex $C^*(A,A)$ of any associative or A_∞-algebra A, i.e. that $\mathrm{Hoch}(A) = C^*(A,A)$ is a 2-algebra.

This can be interpreted precisely as our statement that the deformations of the open string algebra form a consistent (tree level) closed string theory. The Deligne theorem means that there are maps $\mathrm{Hom}(C^*(A,A)^{\otimes n}, C^*(A,A))$ whose compositions are compatible with the ones in the operad of little discs. This is the structure of the open strings we found: the structure of C_2 is formed by the discs with holes, and these have composition laws given by gluing discs in holes. The maps in $\mathrm{Hom}(V^{\otimes n}, V)$ can be seen as a map from a state at the holes of a disc with n holes to a state at the boundary of the disc. Note that one can view the boundary of the disc as a state at infinity. This is the structure of closed string operators. The Deligne theorem can then be interpreted as saying that the element of $\mathrm{Hoch}(A)$ can be inserted at a boundary. These are precisely the states $|\alpha_{a_0}\alpha_{a_1}\cdots\alpha_{a_n}\rangle$, which we should put at the boundary of a hole.

5. Gauge theory realisations

In this sections we study two gauge theory realisations of the above picture, Chern-Simons and holomorphic Chern-Simons theory. They can be viewed as open string theories corresponding to the topological A-model and B-model respectively [22]. The reader should be warned that some of the results of this section are somewhat preliminary, and only a sketch of the structure is presented. More details will be presented in a forthcoming paper [29].

5.1 Chern-Simons theory

Chern-Simons was obtained as the effective field theory for a topological open string theory by Witten [22]. It is the open string sector of the topological A-model, which describes the Kähler moduli space of a Calabi-Yau manifold. The Chern-Simons theory lives on a (special) lagrangian 3-cycle, or supersymmetric D3-brane,[7] embedded in the Calabi-Yau manifold. We know that the theory living on a D-brane is a gauge theory. On this three-cycle lies a flat gauge bundle E, a critical point of CS.

The closed string A-model has operators corresponding to the deformations of the Kähler form and of the B-field on the Calabi-Yau. The BRST cohomology can be identified with the Hodge cohomology $H^{*,*}(M) = H^*(M,\mathbb{C})$ of the Calabi-Yau M. The total BRST operator can be identified with the d-operator.

[7] We work in euclidean space, and count only dimensions lying inside the Calabi-Yau.

The tangent space of the open string model, spanned by the boundary operators α_a, consists of the deformations of the connection A of the gauge bundle E on the 3-cycle, and of the transverse scalars in the adjoint. In the rest we will ignore these scalars, and concentrate on the pure gauge theory. The BRST operator of the open string becomes the covariantised form of the closed string BRST operator, $Q = d_A$. Similarly, the operator G is given by $G = d_A^*$. The derivation condition $Q^2 = 0$ translates to $F = d_A^2 = 0$, expressing the fact that the gauge bundle should be flat. The tangent space is at any point isomorphic to $\Omega^1(M, \mathrm{End}(E))$, the space of adjoint valued 1-forms. We will denote the variations by δA. The corresponding zero-form vertex operator and its descendant are given by $\alpha = \delta A_\mu(X)\chi^\mu$ and $\alpha^{(1)} = \delta A_\mu(X)\dot{X}^\mu$ respectively, where χ^μ denote the fermions that are twisted to worldsheet scalars, and the dot denotes the derivative with respect to a coordinate along the boundary. The BRST cohomology of the open string A-model is therefore given by $H^*(M, \mathrm{End}(E))$.

Using this correspondence of operators and (variation of the) gauge field, we can express all correlators of CS in terms of an effective field theory, living on the worldvolume of the D-brane. The effective action is given by [22]

$$S = \int_M \mathrm{Tr}\Big(\frac{1}{2}A \wedge dA + \frac{1}{3}A \wedge A \wedge A\Big) + \text{instanton corrections}\,. \tag{5.1}$$

The corrections are due to string instantons. Due to these, the theory is not precisely CS, but a deformed one. The deformation parameter is basically the complexified Kähler form. For example, the structure constants of the open string algebra are given in terms of this effective field theory by

$$F_{abc} = \int_M \mathrm{Tr}\Big(\delta_a A \wedge \delta_b A \wedge \delta_c A\Big) + \cdots \tag{5.2}$$

(the dots denote the instanton corrections). The first mixed correlator, for a bulk operator ϕ_i corresponding to a $(1,1)$-form θ_i, is given by

$$\begin{aligned} \Phi_{ia} = \langle \phi_i \alpha_a \rangle &= \langle (\theta_i \chi \tilde{\chi})(\chi \delta_a A(X)) \rangle + \cdots \\ &= \int_M \theta_i \wedge \mathrm{Tr}\Big(\delta_a A\Big) + \cdots\,. \end{aligned} \tag{5.3}$$

Here we used the fact that χ has three zero modes, corresponding to the three longitudinal directions. For the higher mixed correlators, we have to be careful to insert the first descendants $\dot{X}\delta A$ for the extra boundary insertions. The $\dot{X}$ gives extra contributions due to the nonvanishing $X - \dot{X}$ OPE, which are easily calculated. We find for the lower mixed correlators

$$\begin{aligned} \Phi_{iab} &= \int_M \theta_i \wedge \mathrm{Tr}\, d_A^*\Big(\delta_a A \wedge \delta_b A\Big) + \cdots \\ \Phi_{iabc} &= \int_M \theta_i^{\mu\nu}\, \mathrm{Tr}\Big(\partial_\mu \delta_a A \wedge \partial_\nu \delta_b A \wedge \delta_c A\Big) + \cdots\,. \end{aligned} \tag{5.4}$$

The last expression can be seen as the noncommutative variation of the product.

The Hochschild cohomology of the open string algebra $A = H^*(M, \mathrm{End}(E))$ is given by

$$H^*(\mathrm{Hoch}(A)) = H^*(M) \,. \tag{5.5}$$

The important part is the physical deformation, $H^1(\mathrm{Def}(A)) = H^2(\mathrm{Hoch}(A)) = H^2(M)$. The realisation is by the noncommutative deformation of the gauge theory. This seems to be only half of the closed string moduli space, corresponding to the B-field. Comparing to the closed string, for which the ground ring is given by $H^*(M, \mathbb{C})$, the contribution of the Kähler form deformation is missing. It may appear if we study the full supersymmetric theory, including also the sector corresponding to the transverse scalars. Also note that the Kähler form only appears in the instanton corrections, as the pure CS part does not depend on the metric. On the other hand, we considered only the perturbative part of the story here.

The deformation complex is given in more detail in the last column of table 1. First, the symmetries are in $\mathrm{Def}^0(A) = A^0 \oplus A^1 \oplus \mathrm{End}(A)^0$. The first term is the usual gauge symmetry of the gauge field by a zero-form. The second term is the gauge symmetry of the B-field, which can be added in the action to the field strength. The third term is the invariance under reparametrisations of the gauge field.

The physical deformations are given by $\mathrm{Def}^1(A) = A^1 \oplus A^2 \oplus \mathrm{End}(A)^1 \oplus \mathrm{Hom}(A^{\otimes 2}, A)^0$. The first term is the shift of the gauge field. This deformation should be identified with the gauge transformation of the B-field. The second term represents the shift of the field strength by the B-field, which is a deformation of the action. The next term is the deformation of the covariant derivative by (the variation of) the gauge field, which is a map of the algebra (of forms) increasing the degree by one. The last term is the noncommutative deformation of the (wedge) product.

Last, we discuss the obstructions. They are in $\mathrm{Def}^2(A) = A^2 \oplus A^3 \oplus \mathrm{End}(A)^2 \oplus \mathrm{Hom}(A^{\otimes 2}, A)^1 \oplus \mathrm{Hom}(A^{\otimes 3}, A)^0$. The first term corresponds to the condition $d_A \delta A = 0$, which is the equation of motion for the variation of the gauge field. The second one is the Bianchi identity, $d_A F = 0$. The third is the derivation condition on the covariant derivative, $d_A^2 = 0$. Next, we have the Leibniz rule $d_A(fg) = (d_A)g + f d_A g$. The last term then is the associativity of the product, $(f * g) * h = f * (g * h)$. Apart from the first one, they correspond to the constraints in the A_∞ algebra.

5.2 Holomorphic Chern-Simons theory

Holomorphic Chern-Simons theory, also introduced by Witten [22], is the boundary theory related to the closed string B-model, which is described by Kodaira-Spencer theory (KS) – the deformation theory of complex structures on a Calabi-Yau 3-fold M. Much is known about KS (see e.g. [23, 24, 25]), and also HCS has been studied in some detail, though certainly not as much (see e.g. [26, 27, 28]).

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The operators in KS theory are built from the twisted fermions χ_μ and $\eta^{\bar\nu}$. They are sections of the holomorphic tangent bundle T_M and the anti-holomorphic cotangent bundle $\overline{T}^*_M$ to M, hence any element $\theta \in \Gamma(M, \bigwedge^p \overline{T}^*_M \otimes \bigwedge^q T_M)$ corresponds to an operator[8] $\phi = \theta^{\underline{\mu}_p}_{\underline{\bar\nu}_q}(X)\chi_{\underline{\mu}_p}\eta^{\underline{\bar\nu}_q}$. The BRST operator corresponds to the $Q = \bar\partial$ operator on this space, and $G = \bar\partial^*$. The BRST cohomology of KS is given by the spaces $H^{-p,q}(M) \equiv H^q_{\bar\partial}(M, \bigwedge^p T_M) \simeq H^{3-p,q}(M)$. The most important generators, corresponding to the actual moduli of KS, correspond to the forms contained in $H^{-1,1}(M)$.

The open string sector of the B-model is given by holomorphic Chern-Simons theory (HCS) [22]. The KS theory couples naturally to holomorphic bundles, or more generally, to even-dimensional D-branes wrapped around holomorphic cycles in the Calabi-Yau with a holomorphic bundle on it. These are the natural boundary conditions which are invariant under the BRST operator of KS. They naturally describe D6-branes wrapped around the Calabi-Yau space. The BRST operator of the open string becomes the covariantised form of the closed string BRST operator, $Q = \bar\partial_A$, and the operator G is given by $G = \bar\partial_A^*$. The derivation condition $Q^2 = 0$ translates to $F^{0,2} = \bar\partial_A^2 = 0$, expressing the fact that the bundle has to be holomorphic. The tangent space is at any point isomorphic to $\Omega^{0,1}(M, \mathrm{End}(E))$, the space of $(0,1)$-forms with values in the adjoint bundle. The corresponding zero-form vertex operator and its descendant are given by $\alpha = \delta A_{\bar\mu}(X)\eta^{\bar\mu}$ and $\alpha^{(1)} = \delta A_{\bar\mu}(X)\dot{X}^{\bar\mu}$ respectively. The BRST cohomology of the open string B-model is therefore given by $H^*_{\bar\partial_A}(M, \mathrm{End}(E))$.

The effective action is given by [22]

$$S = \int_M \Omega \wedge \mathrm{Tr}\Big(\frac{1}{2} A \wedge \bar\partial A + \frac{1}{3} A \wedge A \wedge A\Big)\,. \tag{5.6}$$

The structure constants of the open string algebra are given in terms of this effective field theory by

$$F_{abc} = \int_M \Omega \wedge \mathrm{Tr}\Big(\delta_a A \wedge \delta_b A \wedge \delta_c A\Big)\,. \tag{5.7}$$

Some mixed correlators, with a $(2,1)$-form μ_i representing the variation of the three form Ω by a change of complex structure, are given by

$$\Phi_{ia} = \int_M \mu_i \wedge \mathrm{Tr}\Big(\delta_a A \wedge F\Big)\,, \tag{5.8}$$

$$\Phi_{iab} = \int_M \mu_i \wedge \mathrm{Tr}\Big(\delta_a A \wedge \partial_A \delta_b A\Big)\,, \tag{5.9}$$

$$\Phi_{iabc} = \int \mu_i \wedge \mathrm{Tr}\Big(\partial_A \delta_a A \wedge \frac{\bar\partial_A^*}{\bar\Delta_A}(\delta_b A \wedge \delta_c A)\Big)\,, \tag{5.10}$$

[8]Underlined indices denote multi-indices.

where $\bar{\Delta}_A = \bar{\partial}_A \bar{\partial}_A^* + \bar{\partial}_A^* \bar{\partial}_A$ is the closed string hamiltonian. As a consistency check, notice that as $\delta_b F^{1,1} = \partial_A \delta_b A$ we can view (5.9) as a variation of (5.8) — in this case there is no ordering problem for the boundary operators. The presence in (5.10) of the single open string propagator, $b_0/L_0 = \bar{\partial}_A^*/\bar{\Delta}_A$, is related to the fact that it is can be calculated using a tree-level calculation of a four-point function, for which we can write down three Feynman diagrams, with a single internal line. Similar formulas can be written down for the higher mixed correlators.

The general mixed correlators form the Hochschild cohomology of HCS. The open string algebra was given by $A = H^*_{\bar{\partial}_A}(M, \mathrm{End}(E))$, generated by the holomorphic coordinates z^μ, and the fermionic zero-modes $\eta^{\bar{\mu}}$. For the closed string algebra, we had $H^{-p,q}(M)$, generated by z^μ, $\eta^{\bar{\mu}}$, and χ_μ. From the mixed correlators we read off the correspondence $H^{q-1}(\mathrm{Hom}(A^{\otimes p+1}, A)) = H^q_{\bar{\partial}}(M, \bigwedge^p T_M)$. This agrees with direct calculation: using the Hochschild-Kostant-Rosenberg theorem, we find that the Hochschild cohomology of A should be generated by the generators of A, supplemented by generators χ_μ of (ghost) degree 1, and $\lambda_{\bar{\mu}}$ of degree zero.[9] In the cohomology, the two degree zero generators do not play much of a role. The two fermionic generators on the other hand precisely generate the bundles T_M and $\overline{T}^*_M$, respectively.

6. Conclusions and outlook

In this paper we have expressed deformations of topological open strings in terms of mixed correlators, identified them with the Hochschild cohomology of the open string algebra, and shown that this Hochschild cohomology is aclosed string algebra. The mixed correlators (3.3) can beregarded as inner products $\Phi_{ia_0a_1\ldots a_n} = \langle i|a_0a_1\ldots a_n\rangle$, intertwining between a boundary state $|a_0a_1\ldots a_n\rangle \equiv |\underline{a}\rangle$ and a bulk state $|i\rangle$. This gives an identification

$$|i\rangle \simeq \sum_{\underline{a}} \Phi_i{}^{\underline{a}} |\underline{a}\rangle\,. \tag{6.1}$$

In the open string channel, open string states can mediate between boundary conditions, such as given by Chan-Paton indices. Denoting the vector space spanned by them by B, the open string state $|\alpha_{\underline{a}}\rangle$ is a single trace in B, so the above relation can be written in terms of operators as

$$\phi_i \simeq \sum_{\underline{a}} \Phi_i{}^{a_0a_1\ldots a_n}\, \mathrm{Tr}\Big(\alpha_{a_0}\alpha^{(1)}_{a_1}\cdots\alpha^{(1)}_{a_n}\Big)\,. \tag{6.2}$$

We have identified the algebraic operations in the closed string theory. The algebraic structure is that of a Gerstenhaber algebra, or in the context of operads of little

[9] In terms of $\mathrm{Hom}(A^{\otimes n}, A)$, one can think of them as the cohomology classes of the maps $\partial/\partial z^\mu$ and $\partial/\partial \eta^{\bar{\mu}}$, respectively.

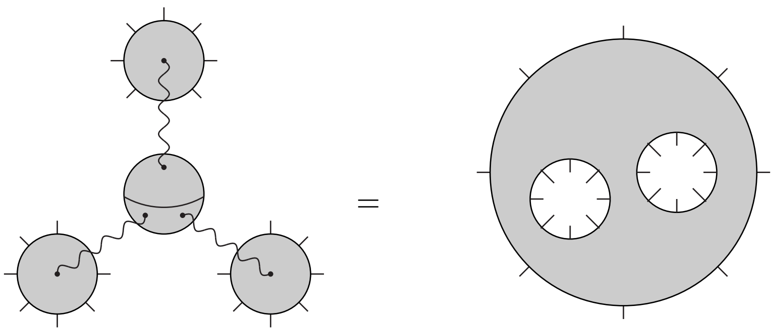

Figure 8: An open string loop diagram.

discs, that of a 2-algebra. This is a physical realisation of the Deligne theorem, and it generalises the Cattaneo-Felder model (function algebras). We have applied our framework to the A- and the B-model.

It would be interesting to investigate if Kontsevich's formality theorem holds for open string algebras, as it does for function algebras. The formality theorem states the existence of a quasi-isomorphism between Hochschild complex and Hochschild cohomology. In the case of function algebras, the essential argument is that the Hochschild cohomology of the Hochschild cohomology is trivial [6], i.e. that the algebra of deformations – or equivalently the closed string — has only trivial deformations itself.[10] In our case it should first be shown that for every infinitesimal deformation of the topological open string there is a closed string state giving rise to this deformation. One expects a full correspondence when the topological open/closed theory is unitary. Subsequently, it should be shown that there are no affine invariant obstructions to the deformation of the deformation algebra. In superstring theory, formality will probably not hold due to the affine invariant obstruction provided by the constant dilaton.

Inclusion of boundary conditions has no effect on the Hochschild cohomology. This is due to the mathematical fact that the Hochschild cohomology is invariant under Morita equivalence. That the closed string theory does not depend on the choice of boundary conditions is in agreement with this.

A further step would be to consider open string loop diagrams, which can also be interpreted as disc diagrams connected by closed strings (see figure 8), or comparing with (6.2), as multi-trace operators. We could view the boundaries with open string insertions as effective closed string operators, using the dictionary (6.1). These generalised boundary operators however will generally not be on-shell. Hence an extension of the deformation theory including off-shell closed string states is necessary. The corresponding objects are homotopy Gerstenhaber or homotopy BV algebras. Furthermore, one can ask the question whether all external closed string insertions can be replaced by these generalised boundary states, or under which conditions this

[10] This seems to contradict the WDVV construction of a deformation of the closed string. However, that is a deformation of the OPE algebra, while the relevant structure for the formality of the Hochschild complex is the bracket, which is not deformed by WDVV.

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can be done. If this is the case, one can calculate all (tree-level) closed string diagrams in terms of (multi-loop) open string diagrams. This is closely related to the open/closed string correspondence.

An obvious generalisation is to higher d-algebras. The structure of a d-algebra is rather similar to that of a perturbative d-dimensional field theory. For higher d, the products are given by the OPE and its higher homotopies, and the brackets are defined by integrating descendants. The generalised Deligne theorem [6, 30] would be a bulk-boundary correspondence; it states the existence of a natural action (in a specific sense) of a $(d+1)$-algebra on a d-algebra. For example, the relation between Floer and Donaldson-Witten theories (see e.g. [31]) can be seen as an action of a 1-algebra (quantum mechanics) on a 0-algebra (the configuration space of instantons). The action of a 3-algebra on a 2-algebra can be studied in a system of open M2-branes ending on M5-branes, where deformations of closed strings might be found. In this case, deformations corresponding to constant 3-form fields seem to correspond to a noncommutative version of loop space [32]–[35]. Work on the algebraic structure of such a system is in progress [36].

Studying the generalisations may provide a new point of view on the AdS/CFT-correspondence (AdS-gravity could be formulated in terms of a d-algebra) and on RG-flow, especially in the context of [37]. Another interesting relation is perhaps [38].

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A. Integrability

In this appendix, we use the Ward identity for the operator G to derive symmetry relations for the mixed correlators. In the presence of a boundary, the symmetry is $\mathrm{SL}(2,\mathbb{R})$ and we have three real vector fields $\xi = \xi(z)\partial_z + \bar{\xi}(\bar{z})\partial_{\bar{z}}$. The Ward identity for G can be written:

$$\begin{aligned} 0 = & \sum_m \xi(x_m)\Big\langle \prod_n \phi_{i_n}(z_n)\alpha_{a_1}(x_1)\cdots\alpha^{(1)}_{a_m}(x_m)\cdots\alpha_{a_r}(x_r)\Big\rangle + \\ & + \sum_n \xi(z_n)\Big\langle \phi_{i_1}(z_1)\cdots\phi^{(1,0)}_{i_n}(z_n)\cdots\phi_{i_s}(z_s)\prod_m \alpha_{a_m}(x_m)\Big\rangle + \\ & + \sum_n \bar{\xi}(\bar{z}_n)\Big\langle \phi_{i_1}(z_1)\cdots\phi^{(0,1)}_{i_n}(\bar{z}_n)\cdots\phi_{i_s}(z_s)\prod_m \alpha_{a_m}(x_m)\Big\rangle . \end{aligned} \tag{A.1}$$

For multiple vector fields and forms, we use the convention that contractions are first applied between the vector field and the form that are closest to each other, e.g. $\xi_2\xi_1\alpha_a\alpha_b \equiv (\xi_1\cdot\alpha_a)(\xi_2\cdot\alpha_b)$. We write $\phi^{(2)} = \phi^{(1,1)}_{z\bar{z}} dz\wedge d\bar{z}$ and note that a contraction with a two-form is antisymmetric,

$$\bar{\xi}_2\xi_1\phi^{(2)} = \bar{\xi}_2\xi_1\phi^{(1,1)}_{z\bar{z}} = -\xi_1\bar{\xi}_2\phi^{(1,1)}_{\bar{z}z} = -\xi_1\bar{\xi}_2\phi^{(2)}\,. \tag{A.2}$$

In the case of one ϕ located in the bulk at w and three α's on the boundary at x_1, x_2 and x_3, we choose two vector fields $\xi_1(z) = (z-x_2)(z-x_3)$ and $\xi_2(z) = (z-x_1)(z-x_3)$. Applying ξ_1,

$$0 = \xi_1(x_1)\langle\phi\alpha^{(1)}\alpha^{(1)}\alpha\rangle + \xi_1(w)\langle\phi^{(1,0)}\alpha\alpha^{(1)}\alpha\rangle + \bar{\xi}_1(\bar{w})\langle\phi^{(0,1)}\alpha\alpha^{(1)}\alpha\rangle\,, \tag{A.3}$$

and then ξ_2, we find

$$\begin{aligned}\xi_2(x_2)\xi_1(x_1)\langle\phi\alpha^{(1)}\alpha^{(1)}\alpha\rangle &= -\xi_2(x_2)\xi_1(w)\langle\phi^{(1,0)}\alpha\alpha^{(1)}\alpha\rangle - \xi_2(x_2)\bar{\xi}_1(\bar{w})\langle\phi^{(0,1)}\alpha\alpha^{(1)}\alpha\rangle\\ &= \bar{\xi}_2(\bar{w})\xi_1(w)\langle\phi^{(2)}\alpha\alpha\alpha\rangle - \xi_2(w)\bar{\xi}_1(\bar{w})\langle\phi^{(2)}\alpha\alpha\alpha\rangle\end{aligned} \tag{A.4}$$

$$\langle\phi\alpha^{(1)}\alpha^{(1)}\alpha\rangle = \Big(\frac{\xi_1(w)}{\xi_1(x_1)}\frac{\bar{\xi}_2(\bar{w})}{\xi_2(x_2)} - \frac{\bar{\xi}_1(\bar{w})}{\xi_1(x_1)}\frac{\xi_2(w)}{\xi_2(x_2)}\Big)\langle\phi^{(2)}\alpha\alpha\alpha\rangle\,. \tag{A.5}$$

Due to the conformal Ward identities, the correlators depend only on the anharmonic ratio $\zeta = [(w-x_1)(x_2-x_3)]/[(w-x_2)(x_1-x_3)]$, which satisfies

$$\xi_i(x_i)\frac{\partial\zeta}{\partial x_i} + \xi_i(w)\frac{\partial\zeta}{\partial w} = 0 \qquad (i=1,2)\,. \tag{A.6}$$

The result can be written as

$$\langle\phi\alpha^{(1)}\alpha^{(1)}\alpha\rangle = \Big(\frac{\partial\zeta}{\partial w}\frac{\partial\bar{\zeta}}{\partial\bar{w}}\Big)^{-1}\Big(\frac{\partial\zeta}{\partial x_1}\frac{\partial\bar{\zeta}}{\partial x_2} - \frac{\partial\zeta}{\partial x_2}\frac{\partial\bar{\zeta}}{\partial x_1}\Big)\langle\phi^{(2)}\alpha\alpha\alpha\rangle. \tag{A.7}$$

Recognising a quotient of jacobians, we conclude

$$\int dx_1dx_2\langle\phi\alpha^{(1)}\alpha^{(1)}\alpha\rangle = \int dwd\bar{w}\langle\phi^{(2)}\alpha\alpha\alpha\rangle\,. \tag{A.8}$$

Next we consider two ϕ's at w and v, and one α at x. We use two vector fields, $\xi_1(z) = z-x$ and $\xi_2(z) = (z-x)^2$. Then we have $(i=1,2)$

$$0 = \xi_i(w)\langle\phi^{(2)}\phi\alpha\rangle + \xi_i(v)\langle\phi^{(0,1)}\phi^{(1,0)}\alpha\rangle + \bar{\xi}_i(\bar{v})\langle\phi^{(0,1)}\phi^{(0,1)}\alpha\rangle \tag{A.9}$$

the solution of which is

$$\begin{pmatrix}\langle\phi^{(0,1)}\phi^{(1,0)}\alpha\rangle\\ \langle\phi^{(0,1)}\phi^{(0,1)}\alpha\rangle\end{pmatrix} = \frac{1}{\xi_1(v)\bar{\xi}_2(\bar{v}) - \bar{\xi}_1(\bar{v})\xi_2(v)}\begin{pmatrix}\bar{\xi}_1(\bar{v})\xi_2(w) - \xi_1(w)\bar{\xi}_2(\bar{v})\\ \xi_1(w)\xi_2(v) - \xi_1(v)\xi_2(w)\end{pmatrix}\langle\phi^{(2)}\phi\alpha\rangle. \tag{A.10}$$

Interchanging w and v yields as the second equation,

$$\langle \phi^{(0,1)}\phi^{(0,1)}\alpha\rangle = \frac{\xi_1(v)\xi_2(w) - \xi_1(w)\xi_2(v)}{\xi_1(w)\bar{\xi}_2(\bar{w}) - \bar{\xi}_1(\bar{w})\xi_2(w)}\langle \phi\phi^{(2)}\alpha\rangle\,, \tag{A.11}$$

hence

$$\langle \phi^{(2)}\phi\alpha\rangle = -\frac{\xi_1(v)\bar{\xi}_2(\bar{v}) - \bar{\xi}_1(\bar{v})\xi_2(v)}{\xi_1(w)\bar{\xi}_2(\bar{w}) - \bar{\xi}_1(\bar{w})\xi_2(w)}\langle \phi\phi^{(2)}\alpha\rangle\,. \tag{A.12}$$

Using the anharmonic ratio $\zeta = \frac{(v-x)(w-\bar{w})}{(w-x)(v-\bar{w})}$, satisfying similar relations as the one above, we can write

$$\langle \phi^{(2)}\phi\alpha\rangle = \Big(\frac{\partial\zeta}{\partial v}\frac{\partial\bar{\zeta}}{\partial\bar{v}} - \frac{\partial\zeta}{\partial\bar{v}}\frac{\partial\bar{\zeta}}{\partial v}\Big)^{-1}\Big(\frac{\partial\zeta}{\partial w}\frac{\partial\bar{\zeta}}{\partial\bar{w}} - \frac{\partial\zeta}{\partial\bar{w}}\frac{\partial\bar{\zeta}}{\partial w}\Big)\langle \phi\phi^{(2)}\alpha\rangle\,, \tag{A.13}$$

so that

$$\int dw d\bar{w}\langle \phi^{(2)}\phi\alpha\rangle = \int dv d\bar{v}\langle \phi\phi^{(2)}\alpha\rangle\,. \tag{A.14}$$

All these derivations remain true when there are extra insertions of $\int\phi^{(2)}$ and $\int\alpha^{(1)}$.

B. Algebras up to homotopy

A *homotopy associative* or A_∞*-algebra* can be defined in terms of a derivation d acting on the tensor algebra $\mathcal{T}A = \bigoplus_{n\geq 0} A^{\otimes n}$ of a (graded) vector space A. The derivation is completely determined by the map from $\mathcal{T}A$ to A. We denote the component of d mapping the nth tensor product $A^{\otimes n}$ to A by d_n. So we have $d = d_1 + d_2 + d_3 + \cdots$. All d_k are derivations in the sense that

$$d_k(a_1,\ldots,a_{k+n}) = \sum_{i=0}^{n}(-1)^{i(n-1)}(a_1,\ldots,d_k(a_{i+1},\ldots,a_{i+k}),\ldots,a_{k+n}). \tag{B.1}$$

Furthermore, d is a twisted differential, in the following sense. Considering the shifted algebra $sA = A[-1]$, called the suspension.[11] The shifted maps $b_k = s\circ d_k\circ(s^{-1})^{\otimes k}$ should form a differential on the suspension, i.e. $b^2 = 0$, of degree -1. This implies an infinite number of homogeneous relations for the d_k; for any $n \geq 1$,

$$\sum_{k+l=n+1}(-1)^{(k-1)l}d_k\circ d_l = 0\,. \tag{B.2}$$

The map d_k has parity $k \bmod 2\mathbb{Z}$. Explicitly, the first few relations read $d_1^2 = 0$, $d_1d_2 = d_2d_1$, $d_2^2 = -d_1d_3 - d_3d_1$, $d_2d_3 - d_3d_2 = -d_1d_4 - d_4d_1$. These say that d_1 is a differential on A, d_2 is a product for which d_1 is a derivation, d_3 gives a correction to

[11] For an integer k, $[k]$ denotes a shift of the degree of a complex $C = \bigoplus_n C^n$ by k, that is $C[k]^n := C^{k+n}$; therefore, $sC^n = C^{n-1}$. Physically, the suspension corresponds to descent.

the associativity of this product (d_2^2 is the associator), etc. We often write $d_1(a) = \delta a$ and $d_2(a_1, a_2) = a_1 * a_2$.

Homotopy Lie or L_∞*-algebras* are defined in a similar way. We also start with a (graded) space A. The only difference is that everything should be (graded) anti-symmetric. For example, the tensor product of the algebra is replaced by the (graded) exterior product, $\bigoplus_n \bigwedge^n A$, and the products d_n are all (graded) anti-commutative. They are called brackets, and $d_2^2 = 0$ is the Jacobi identity for the Lie bracket.

A *Gerstenhaber algebra* (G-algebra) is a $\mathbb{Z}$-graded algebra with a graded commutative associative product $\cdot$ of degree 0 and a bracket $[\cdot,\cdot]$ of degree -1 (the Gerstenhaber bracket), which is such that $A[-1]$ is a graded Lie algebra. The operations satisfy some relations. The obvious ones are the Leibniz rules for the product and the bracket, graded associativity for the product and graded Jacobi for the bracket. Furthermore, for any fixed element α, the map $\beta \mapsto [\alpha, \beta]$ must be a graded derivation of the product.

A *homotopy Gerstenhaber algebra* (G_∞-algebra), is defined similarly to A_∞ by first introducing a derivation/differential d and then relaxing both associativity conditions. A G_∞-algebra has contained in it both a commutative A_∞ and a L_∞-algebra, according to the scheme

$$\begin{array}{ccccc} G_\infty & & \to & L_\infty & \\ & d & [\cdot,\cdot] & [\cdot,\cdot,\cdot] & \cdots \\ \downarrow & \cdot & & & \\ \text{comm. } A_\infty & \{\cdot,\cdot,\cdot\} & & & \\ & \vdots & & & \end{array} . \tag{B.3}$$

Horizontally, we have the Gerstenhaber bracket and the corresponding higher bracket forming the L_∞ structure, while vertically we see the A_∞ structure, based on the commutative product and the higher products. They share the same d, so that on the d-cohomology one always gets a Gerstenhaber algebra. An operad definition of a homotopy Gerstenhaber algebra can be found in [8].

References

[1] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [`hep-th/9711162`].

[2] A.S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantisation formula*, `math.QA/9902090`.

[3] V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [`hep-th/9903205`].

[4] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [`hep-th/9908142`].

JHEP01(2001)035

[5] M. Kontsevich, *Deformation quantisation of Poisson manifolds, I*, [q-alg/9709040].

[6] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[7] B. Zwiebach, *Closed string field theory: quantum action and the b-v master equation*, *Nucl. Phys.* **B 390** (1993) 33 [hep-th/9206084].

[8] T. Kimura, A.A. Voronov, G.J. Zuckerman, *Homotopy Gerstenhaber Algebras and Topological Field Theory*, [q-alg/9602009].

[9] T. Kimura, J. Stasheff and A.A. Voronov, *On operad structures of moduli spaces and string theory*, *Comm. Math. Phys.* **171** (1995) 1 [hep-th/9307114].

[10] J. Stasheff, *Closed string field theory, strong homotopy Lie algebras and the operad actions of moduli spaces*, hep-th/9304061.

[11] B.H. Lian and G.J. Zuckerman, *Algebraic and geometric structures in string backgrounds*, hep-th/9506210.

[12] E. Witten and B. Zwiebach, *Algebraic structures and differential geometry in* 2*-D string theory*, *Nucl. Phys.* **B 377** (1992) 55 [hep-th/9201056].

[13] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Topological strings in $d < 1$*, *Nucl. Phys.* **B 352** (1991) 59.

[14] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, *Surv. Diff. Geom.*, **1** (1991) 243.

[15] B. Zwiebach, *Oriented open-closed string theory revisited*, *Annals Phys.* **267** (1998) 193 [hep-th/9705241].

[16] M. Gerstenhaber and S.D. Schack, *Algebras, bialgebras, quantum groups and alebraic deformations*, *Cont. Math.* , **134** (1992) 51.

[17] J. Stasheff, *Homotopy associativity of H-spaces*, I and II, *Trans. Amer. Math. Soc.* , **108** (1963) 275.

[18] E. Witten, *On background independent open string field theory*, *Phys. Rev.* **D 46** (1992) 5467 [hep-th/9208027].

[19] M. Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, 1990-1992, Birkhäuser 1993 p. 173.

[20] E. Verlinde, *The master equation of 2-d string theory*, *Nucl. Phys.* **B 381** (1992) 141 [hep-th/9202021].

[21] D. Tamarkin, *Another proof of M. Kontsevich formality theorem*, math.QA/9803025.

[22] E. Witten, *Chern-Simons gauge theory as a string theory*, hep-th/9207094.

[23] K. Kodaira and D.C. Spencer, *On deformations of complex analytic structures, 1, 2*, *Annals of Math.* II **67** (1958) 45.

[24] E. Witten, *Mirror manifolds and topological field theory*, in *Mirror Symmetry I*, S.T. Yau ed., International Press (1992) [hep-th/9112056].

[25] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Comm. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[26] R. Gopakumar and C. Vafa, *Topological gravity as large-N topological gauge theory*, *Adv. Theor. Math. Phys.* **2** (1998) 413 [hep-th/9802016].

[27] C. Vafa, *Extending mirror conjecture to Calabi-Yau with bundles*, [hep-th/9804131].

[28] C. Hofman and J.-S. Park, *Sigma models for bundles on Calabi-Yau: a proposal for matrix string compactifications*, *Nucl. Phys.* **B 561** (1999) 125 [hep-th/9904150].

[29] C. Hofman and W.K. Ma, *Kodaira-Spencer as a deformation of holomorphic Chern-Simons*, in preparation.

[30] M. Kontsevich, Y. Soibelman *Deformations of algebras over operads and Deligne's conjecture* [math.QA/0001151].

[31] E. Witten, *Topological quantum filed theory*, *Comm. Math. Phys.* **117** (1988) 353.

[32] E. Bergshoeff, D.S. Berman, J.P. van der Schaar and P. Sundell, *A noncommutative M-theory five-brane*, *Nucl. Phys.* **B 590** (2000) 173 [hep-th/0005026].

[33] E. Bergshoeff, D.S. Berman, J.P. van der Schaar and P. Sundell, *Critical fields on the m5-brane and noncommutative open strings*, *Phys. Lett.* **B 492** (2000) 193 [hep-th/0006112].

[34] S. Kawamoto and N. Sasakura, *Open membranes in a constant C-field background and noncommutative boundary strings*, *J. High Energy Phys.* **07** (2000) 014 [hep-th/0005123].

[35] R. Gopakumar, S. Minwalla, N. Seiberg and A. Strominger, *Om theory in diverse dimensions*, *J. High Energy Phys.* **08** (2000) 008 [hep-th/0006062].

[36] C. Hofman and W.K. Ma, *Topological Model of M5-branes in a Background C-Field*, in preparation.

[37] J. Khoury and H. Verlinde, *On open/closed string duality*, hep-th/0001056.

[38] E. Witten, *Duality relations among topological effects in string theory*, *J. High Energy Phys.* **05** (2000) 031 [hep-th/9912086].

JHEP01(2001)035